Int. J. Heat Mass Transfer. Vol. 22, pp. 1347–1353
Pergamon Press Ltd. 1979. Printed in Great Britain

FINITE ELEMENT ANALYSIS OF OPTIMAL HEATING OF A SLAB WITH TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY

R. A. MERIC
Marmara Scientific and Industrial Research Institute, Applied Mathematics Division, Gebze, Kocaeli, Turkey

(*Received* 13 *March* 1978 *and in revised form* 18 *December* 1978)

Abstract—An infinitely long solid slab of temperature dependent thermal conductivity is heated optimally by the ambient temperature. The physical objective of the problem is to increase the temperature level in the slab to a higher level at the end of a fixed period of time, while keeping the necessary ambient temperature level as low as possible. Reformulated as an optimal control problem, the problem is solved numerically by utilizing the finite element method. An approximate perturbation method is also given for linearizing the necessary conditions for optimality.

NOMENCLATURE

- Bi, Biot number;
- J, performance index defined by equation (6);
- J', functional defined by equation (17);
- k, thermal conductivity of the slab;
- N, shape function;
- T, temperature in the slab;
- t, time coordinate;
- u, controlling ambient temperature;
- x, spatial coordinate along the plate thickness.

Greek symbols

- α, weighting parameter;
- Δ, increment;
- δ, variational symbol;
- ε, slope of thermal conductivity–temperature curve; perturbation parameter;
- λ, Lagrange multiplier function.

Subscripts

- d, desired state;
- f, final time;
- i, initial time;
- j, dummy index;
- t, partial differentiation with respect to t;
- x, partial differentiation with respect to x;
- 0, 0th order perturbation;
- 1, 1st order perturbation.

1. INTRODUCTION

THERE are many industrial processes in which it is required to control the temperature distribution in a given material. Such a situation arises in the glass industry, for example, where glass nearing the final stages of certain manufacturing processes must be brought to a temperature as near uniform as possible to prevent unwanted inhomogeneities in the final article [1].

In iron and steel industry, on the other hand, in heating ingots before rolling it is important to heat the metal rapidly and sufficiently uniformly. A rapid heating of the metal increases the productive capacity of furnaces, while uniformly heated ingots insure less defective products.

In this investigation, optimal heating of an infinitely long solid slab with temperature dependent thermal conductivity is analyzed numerically. The posed problem mathematically constitutes a so-called optimal boundary control problem [2]. After obtaining the necessary conditions for optimality by calculus of variations, finite element methods are utilized for numerical solutions.

2. STATEMENT OF THE PROBLEM

An infinitely long undeformable solid slab is required to be heated from a given initial temperature level to a higher desired level by boundary convection in a fixed period of time. The thermal conductivity of the slab happens to obey a linear thermal conductivity–temperature relationship. The one-dimensional differential equation which characterizes the heat conduction in the slab may be written in a nondimensional form as

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right], \quad 0 \leqslant x \leqslant 1, \quad 0 < t \leqslant t_f, \qquad (1)$$

and

$$k(T) = 1 + \varepsilon T. \qquad (2)$$

Here $T(x,t)$ represents the temperature distribution in the slab whose thickness is equal to 1. The slope of the dimensionless conductivity–temperature curve is taken as ε. The optimal (controlled) heating of the slab is assumed to take place from the initial time $t = 0$ to the final time $t = t_f$.

The initial and boundary conditions of the problem may be prescribed as

$$t = 0;\ T = T_i, \tag{3}$$

$$x = 0;\ \frac{\partial T}{\partial x} = 0, \tag{4}$$

$$x = 1;\ k\frac{\partial T}{\partial x} = Bi(u - T), \tag{5}$$

where T_i is the constant initial temperature level in the slab; Bi is the usual Biot number indicating the ratio of the surface conductance to the conduction of solid; $u(t)$ is the controlling ambient temperature whose functional dependence on time is sought. The boundary conditions (4) and (5) represent the insulation condition at one face of the slab and the convection condition at the other, respectively. In equation (5), Bi should be considered as an equivalent Biot number, the heat being transmitted to the slab by a combined effect of conduction, convection and radiation.

The physical objective of the control problem is to increase the temperature level of the slab from $T = T_i$ to as near as $T = T_d$ at the end of a given final time t_f, where T_d is the desired temperature level. However, the problem would not be well-posed without some form of constraint on the control function, i.e., the ambient temperature [1]. This constraint may be taken as forcing the ambient temperature as near as zero, while achieving a final-time temperature as near as T_d. Thus, the stated objectives of the problem may be cast into a mathematical form as indicated by a performance index J:

$$J = \frac{1}{2}\int_0^1 (T - T_d)^2\Big|_{t = t_f} \mathrm{d}x + \frac{\alpha}{2}\int_0^{t_f} u^2 \mathrm{d}t, \tag{6}$$

where α is a given weighting parameter. The first term in the above quadratic functional is the spacewise integral of the square of the deviation of the final-time temperature from the desired temperature level over the thickness of the slab. The second term is, on the other hand, the timewise integration of the square of the ambient temperature over the control time t_f with a weighting coefficient. The physical objectives of the problem are attained with a relative degree of achievement according to the value of α when the performance index J is minimized. Hence, the function $u(t)$ which minimizes J is the desired optimal ambient temperature solution.

The weighting parameter α plays an important role in the problem. It combines actually two physical objectives in a linear combination by weighting. At this point it can be argued that taking a smaller value for α would result in final-time temperatures nearer to the desired level. Nevertheless, if the fuel cost (related to the ambient temperature) is relatively important one might not choose a very small value for α.

3. METHOD OF SOLUTION

3.1. *Necessary conditions for optimality*

The posed optimal control problem is to find the ambient temperature $u(t)$ which minimizes the performance index J while the heat conduction equation, and the initial and boundary conditions are satisfied. However, instead of treating the problem as the minimization of a functional subject to equality constraints, we may obtain the necessary conditions for optimality, which are only a set of partial differential equations, by calculus of variations [2]. The solution of these differential equations then represents the desired optimal ambient temperature as well as the temperature distribution in the slab. Thus the nonlinear necessary conditions for optimality may be stated as follows [2]:

$$\frac{\partial T}{\partial t} = (1 + \varepsilon T)\frac{\partial^2 T}{\partial x^2} + \varepsilon\left(\frac{\partial T}{\partial x}\right)^2, \tag{7}$$

$$\frac{\partial \lambda}{\partial t} = -(1 + \varepsilon T)\frac{\partial^2 \lambda}{\partial x^2}, \tag{8}$$

together with the initial time condition

$$t = 0;\ T = T_i, \tag{9}$$

the final time condition

$$t = t_f;\ \lambda = T - T_d, \tag{10}$$

and the boundary conditions

$$x = 0;\ \frac{\partial T}{\partial x} = 0, \tag{11}$$

$$x = 0;\ \frac{\partial \lambda}{\partial x} = 0, \tag{12}$$

$$x = 1;\ (1 + \varepsilon T)\frac{\partial T}{\partial x} = Bi(u - T) \tag{13}$$

$$x = 1;\ (1 + \varepsilon T)\frac{\partial \lambda}{\partial x} + Bi\lambda = 0, \tag{14}$$

$$x = 1;\ \alpha u + Bi\lambda = 0, \tag{15}$$

where $\lambda = \lambda(x, t)$ is the Lagrangian multiplier of calculus of variations.

The optimality conditions (7) through (15) involve 3 solution functions, namely $u(t)$, $T(x,t)$ and $\lambda(x,t)$. The partial differential equations have a two-point boundary value character in time as well as in the space variable. In other words, no complete time-conditions are available at either the initial or the final time. This in turn makes the solution of this type of problem even numerically quite difficult.

3.2. *Finite element method for the optimality conditions*

It is impossible to find a closed form solution to the nonlinear set of partial differential equations which describes the optimality conditions. Instead, a numerical procedure is to be adopted by using the well-known finite element method [3, 4].

First, equations (7) through (15) may be put into a stationary variational formulation by considering the following [5]:

$$\delta J' = \int_0^{t_f}\int_0^1 [(T_{xx}+\varepsilon T T_{xx}+\varepsilon T_x^2 - T_t)\delta\lambda + (\lambda_{xx}+\varepsilon T\lambda_{xx}+\lambda_t)\delta T]\,\mathrm{d}x\mathrm{d}t - \int_0^{t_f}\left[\left(T_x+\varepsilon T T_x + Bi\,T + Bi^2\frac{\lambda}{\alpha}\right)\delta\lambda + (\lambda_x+\varepsilon T\lambda_x + Bi\lambda)\delta T\right]_{x=1}\mathrm{d}t - \int_0^1 (\lambda - T + T_d)\delta T|_{t=t_f}\,\mathrm{d}x, \qquad (16)$$

where the subscripts x and t denote the partial differentiations with respect to the corresponding independent variables, and the symbol δ designates the first variation. Using integration by parts and reordering some of the terms it can be shown that the optimal functions make the following functional J' stationary:

$$J' = \int_0^{t_f}\int_0^1 [(1+\varepsilon T)T_x\lambda_x + \lambda T_t]\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^{t_f}\left[Bi^2\frac{\lambda^2}{\alpha} + 2\,Bi\,T\lambda\right]_{x=1}\mathrm{d}t + \frac{1}{2}\int_0^1 [-T^2+2T_dT]_{t=t_f}\,\mathrm{d}x. \qquad (17)$$

The optimal control problem is thus reformulated as a stationary variational principle rather than an extremum variational principle. The finite element method may now be applied to this control problem [3]. In the formulation, not only the temperature distribution T, but also the Lagrangian function are interpolated continuously over the finite elements.

The solution domain $(0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant t_f)$ is first divided into triangular elements in space and time as in Fig. 1. Linear interpolation functions are sufficient for compatibility, thus over each element

$$T(x,t) = \sum_{j=1}^{3} N_j(x,t)T_j, \qquad (18)$$

$$\lambda(x,t) = \sum_{j=1}^{3} N_j(x,t)\lambda_j, \qquad (19)$$

where N_j are the usual shape functions defined piecewise, element by element, and are linear herein. The subscript j here denotes the j'th node of an element.

Applying the finite element discretizations to the functional J', equation (17), and making it stationary with respect to the unknown nodal values would result in nonlinear algebraic matrix equations. As the algebraic equations constitute a nonlinear system, an iterative scheme of solution is usually required. In this analysis, the Newton–Raphson method is adopted for the numerical solution of equations [6]. The solution of the matrix equations gives the nodal values of the temperature distribution function and the Lagrangian multiplier function, and thereby those of the ambient temperature, as well.

Given ε is small, it is also possible to use a perturbation scheme to linearize the nonlinear optimality conditions first, and then use the finite element procedure. Thus, the resulting linear algebraic equations can easily be solved by direct methods, such as a Gaussian elimination technique. The perturbation equations and the corresponding functionals are given in the Appendix.

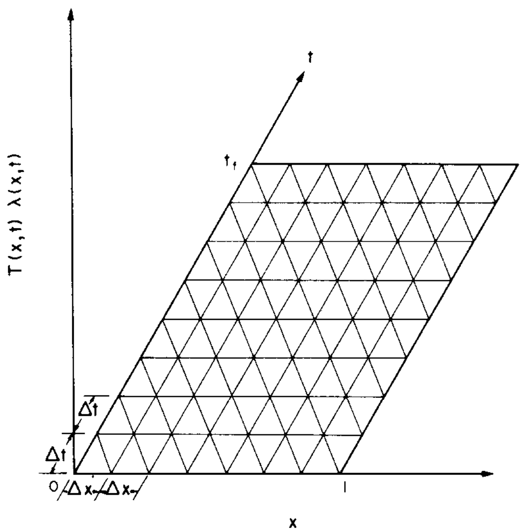

FIG. 1. Solution domain divided into triangular elements in space and time.

4. NUMERICAL RESULTS AND CONCLUSIONS

Numerical results are presented for the optimal heating of an infinitely long slab of temperature dependent conductivity. The objective of the control problem is to have a final-time temperature in the slab as near a desired level with the least amount of fuel cost, represented here by the ambient temperature. In reality, there is a weighting factor α which sets the relative degree of importance of the fuel cost versus the final-time temperature level in the slab.

In Fig. 2, the numerical solutions of the performance index J are plotted against the "nonlinearity" parameter ε for a set of problem parameters. The solid line gives the nonlinear (Newton–Raphson) solution, while the perturbation solution is indicated by a broken line. For small absolute values of ε (i.e., $|\varepsilon| \leqslant 0.2$) the numerical perturbation method gives satisfactory results for this slightly nonlinear problem. For higher $|\varepsilon|$, on the other hand, the reason for increasing discrepancy is probably that the calculation by the perturbation method proceeds by the accumulation of the linearized solutions.

In the nonlinear method of solution, in order to solve the nonlinear simultaneous equations by the Newton–Raphson method, large sets of symmetric linear equations are solved repeatedly. However, as the numerical solution with $\varepsilon = 0$ is taken as the

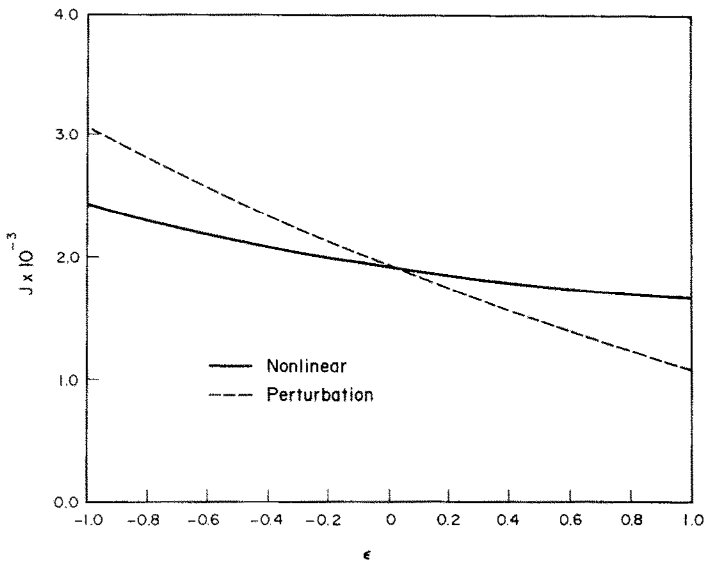

FIG. 2. Performance index J as calculated by two methods for $Bi = 1$; $\alpha = 0.01$; $t_f = 0.4$; $T_i = 0$ and $T_d = 0.2$.

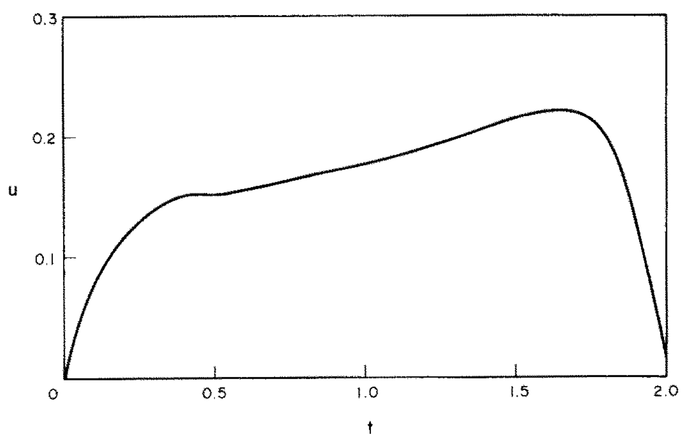

FIG. 3. Ambient temperature $u(t)$ for $\varepsilon = 0.1$; $Bi = 0.5$; $\alpha = 0.001$; $t_f = 2$; $T_i = 0$ and $T_d = 0.2$.

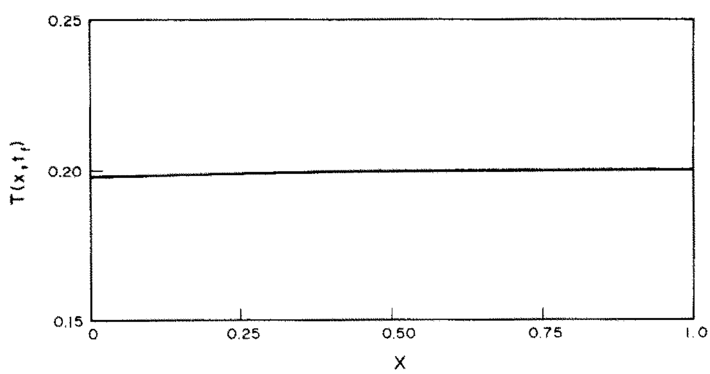

FIG. 4. Final-time temperature $T(x, t_f)$ for $\varepsilon = 0.1$; $Bi = 0.5$; $\alpha = 0.001$; $t_f = 2$; $T_i = 0$ and $T_d = 0.2$.

initial guess, the method converges rapidly: after approximately 6 iterations, the largest residual is 10^{-7}.

In Fig. 3, the optimal ambient temperature $u(t)$ is shown as a function of time when the problem parameters are set as $\varepsilon = 0.1$; $Bi = 0.5$; $\alpha = 0.001$; $t_f = 2$; $T_i = 0$ and $T_d = 0.2$. The corresponding final-time temperature distribution in the slab is given as a function of x in Fig. 4. In the above set of parameters, the weighting parameter α is taken rather small. This indicates a relative unimportance of the fuel cost (the ambient temperature), and in turn results in the final-time temperature very close to the desired level.

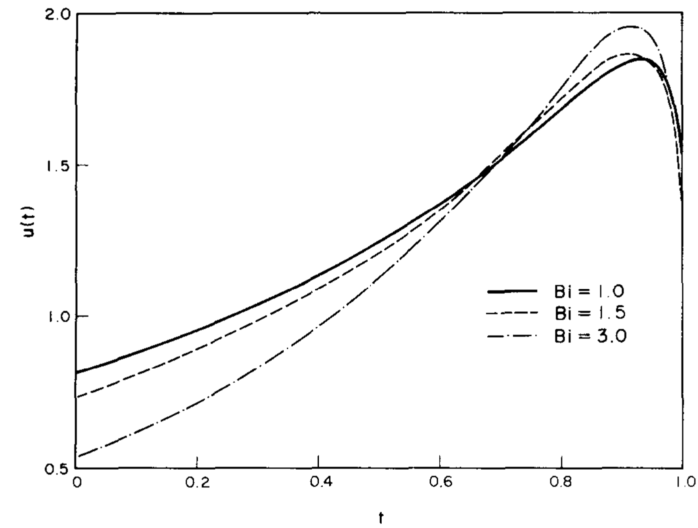

FIG. 5. Ambient temperature $u(t)$ with various Bi for $\varepsilon = 0.2$; $\alpha = 0.125$; $t_f = 1$; $T_i = 1$ and $T_d = 1.5$.

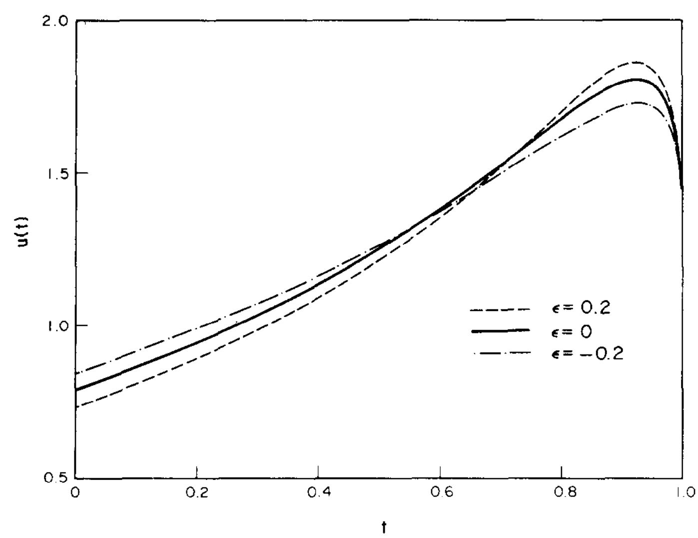

FIG. 6. Ambient temperature $u(t)$ with various ε for $Bi = 1.5$; $\alpha = 0.125$; $t_f = 1$; $T_i = 1$ and $T_d = 1.5$.

In order to see the influence of the Biot number Bi on the optimal ambient temperature, various Bi values have been taken while keeping $\varepsilon = 0.2$; $\alpha = 0.125$; $t_f = 1$; $T_i = 1$ and $T_d = 1.5$. The resulting numerical solutions are shown in Fig. 5.

Finally, in Fig. 6 the numerical solution of the ambient temperature is shown as a function of time for various values of ε when $Bi = 1.5$; $\alpha = 0.125$; $t_f = 1$; $T_i = 1$ and $T_d = 1.5$. From the figure it is seen that for small t the ambient temperature requirement is smaller for a negative ε than the one for a positive ε. It might be argued that this is because a negative thermal conductivity-temperature slope suppresses heat condition in the slab while a positive one augments it.

In the numerical calculations, the influence of the discretization in the space domain $(0 \leqslant x \leqslant 1)$ seems to be modest. Nevertheless, much more finite time discretizations are necessary for numerical stability.

In the analysis, the temperature dependence of thermal conductivity is directly incorporated into the stationary variational formulations. If this relationship should be other than the simple linear form used, this would result in additional algebraic operations because of the higher terms. However, the basic mathematical operations would remain unchanged [7].

As a summary, the optimal heating of an infinitely long slab is analyzed by the finite element method. In the analysis, the time coordinate is treated as if it were another space coordinate. In other words, the problem is considered as a two-point boundary value problem with a closed domain of solution $(0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant t_f)$. With this view, the application of the finite element method becomes very useful as standard programs should easily be adapted for the suggested numerical procedure [5]. At this point it might be pointed out that a solution could also be achieved by using finite differences (possibly in a variational form). However, a two-or three- dimensional problem with a complex geometry would be more amenable to finite element methods.

REFERENCES

1. D. E. Bourne, E. Huntley, I. J. Hendrie and B. P. Hunter, Temperature control of Poiseuille flow with particular application to flow of molten glass, *Int. J. Heat Mass Transfer* **19**, 1045–1055 (1976).
2. A. P. Sage, *Optimum Systems Control*. Prentice-Hall, Inc., Englewood Cliffs, N.J. (1968).
3. O. E. Zienkiewicz, *The Finite Element Method in Engineering Science*. McGraw-Hill, London (1971).
4. G. Gomini and R. W. Lewis, A numerical solution of two-dimensional problems involving heat and mass transfer, *Int. J. Heat Mass Transfer* **19**, 1387–1392 (1976).
5. A. L. Nayak and P. Cheng, Finite element analysis of laminar convective heat transfer in vertical ducts with arbitrary cross-sections, *Int. J. Heat Mass Transfer* **18**, 227–236 (1975).
6. G. D. Byrne and C. A. Hall, *Numerical Solution of Systems of Nonlinear Algebraic Equations*. Academic Press, New York (1973).
7. D. F. Hayes and H. N. Curd, Heat conduction in solids: Temperature dependent thermal conductivity, *Int. J. Heat Mass Transfer* **11**, 285–295 (1968).
8. A. Aziz and J. Y. Benzies, Application of perturbation techniques to heat transfer problems with variable thermal properties, *Int. J. Heat Mass Transfer* **19**, 271–276 (1976).

APPENDIX

For most materials and applications, the thermal conductivity–temperature slope ε is small. Therefore, in order to linearize the optimality conditions of the problem, a regular first order asymptotic expansion for T, λ and u in the perturbation parameter ε may be taken [8]

$$T = T_0 + \varepsilon T_1 + O(\varepsilon^2), \tag{A.1}$$

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + O(\varepsilon^2), \tag{A.2}$$

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2), \tag{A.3}$$

where the subscripts refer to the zeroth and first order solutions. Introducing equations (A.1) through (A.3) into equations (7) through (15), and equating the coefficients of the same order terms in ε, the following linear simultaneous equation systems can be obtained:

for the Oth order term:

$$\frac{\partial T_0}{\partial t} = \frac{\partial^2 T_0}{\partial x^2}, \tag{A.4}$$

$$\frac{\partial \lambda_0}{\partial t} = -\frac{\partial^2 \lambda_0}{\partial x^2}, \tag{A.5}$$

$$t = 0;\ T_0 = T_i, \tag{A.6}$$

$$t = t_f;\ \lambda_0 = T_0 - T_d, \tag{A.7}$$

$$x = 0;\ \frac{\partial T_0}{\partial x} = 0, \tag{A.8}$$

$$x = 0;\ \frac{\partial \lambda_0}{\partial x} = 0, \tag{A.9}$$

$$x = 1;\ \frac{\partial T_0}{\partial x} = Bi\,(u_0 - T_0), \tag{A.10}$$

$$x = 1;\ \frac{\partial \lambda_0}{\partial x} + Bi\,\lambda_0 = 0, \tag{A.11}$$

$$x = 1;\ \alpha u_0 + Bi\,\lambda_0 = 0. \tag{A.12}$$

for the 1st order term:

$$\begin{aligned}\frac{\partial T_1}{\partial t} &= \frac{\partial^2 T_1}{\partial x^2} + T_0\frac{\partial^2 T_0}{\partial x^2} + \left(\frac{\partial T_0}{\partial x}\right)^2 \\ &= \frac{\partial^2 T_1}{\partial x^2} + T_0\frac{\partial T_0}{\partial t} + \left(\frac{\partial T_0}{\partial x}\right)^2,\end{aligned} \tag{A.13}$$

$$\frac{\partial \lambda_1}{\partial t} = -\frac{\partial^2 \lambda_1}{\partial x^2} - T_0\frac{\partial^2 \lambda_0}{\partial x^2} = -\frac{\partial^2 \lambda_1}{\partial x^2} + T_0\frac{\partial \lambda_0}{\partial t}, \tag{A.14}$$

$$t = 0;\ T_1 = 0, \tag{A.15}$$

$$t = t_f;\ \lambda_1 = T_1, \tag{A.16}$$

$$x = 0;\ \frac{\partial T_1}{\partial x} = 0, \tag{A.17}$$

$$x = 0;\ \frac{\partial \lambda_1}{\partial x} = 0, \tag{A.18}$$

$$x = 1;\ \frac{\partial T_1}{\partial x} + T_0\frac{\partial T_0}{\partial x} = Bi\,(u_1 - T_1), \tag{A.19}$$

$$x = 1;\ \frac{\partial \lambda_1}{\partial x} + T_0\frac{\partial \lambda_0}{\partial x} + Bi\,\lambda_1 = 0, \tag{A.20}$$

$$x = 1;\ \alpha u_1 + Bi\,\lambda_1 = 0. \tag{A.21}$$

As can be noticed, the 1st order equations involve the prior solution functions, i.e., the 0th order functions, as the nonhomogeneous terms.

The 0th and 1st order perturbation equations may be put into stationary variational forms as in the nonlinear case. These variational formulations lead to the following functionals:

$$\begin{aligned}J_0 = &\int_0^{t_f}\int_0^1 (T_{0_x}\lambda_{0_x} + \lambda_0\lambda_{0_t})\,\mathrm{d}x\,\mathrm{d}t \\ &+ \frac{1}{2}\int_0^{t_f}\left[Bi^2\frac{\lambda_0^2}{\alpha} + 2Bi\,T_0\lambda_0\right]_{x=1}\mathrm{d}t \\ &+ \frac{1}{2}\int_0^1 [-T_0^2 + 2T_d T_0]_{t=t_f}\,\mathrm{d}x,\end{aligned} \tag{A.22}$$

and

$$\begin{aligned}J_1 = &\int_0^{t_f}\int_0^1 [T_{1_x}\lambda_{1_x} + \lambda_1 T_{1_t} + T_0\lambda_{0_t}T_1 \\ &- (T_0 T_{0_t} + T_{0_x}^2)\lambda_1]\,\mathrm{d}x\,\mathrm{d}t \\ &+ \frac{1}{2}\int_0^{t_f}\left[Bi^2\frac{\lambda_1^2}{\alpha} + 2Bi\,T_1\lambda_1 + 2T_0\lambda_{0_x}T_1 + 2T_0T_{0_x}\lambda_1\right]_{x=1}\mathrm{d}t \\ &- \frac{1}{2}\int_0^1 T_1^2\Big|_{t=t_f}\,\mathrm{d}x.\end{aligned} \tag{A.23}$$

Applying the finite element discretizations (18) and (19) directly to the functionals J_0 and J_1 will give systems of linear algebraic equations which are solvable, for example, by a Gaussian elimination technique. Thus, a direct solution of algebraic equations is involved instead of an iterational solution with its inherent problem of convergence.

ANALYSE PAR ELEMENTS FINIS DU CHAUFFAGE OPTIMAL D'UN LINGOT A CONDUCTIVITE THERMIQUE VARIABLE AVEC LA TEMPERATURE

Résumé—Un lingot infiniment long et à conductivité thermique variable avec la température est chauffé de façon optimale par l'ambiance. L'objectif du problème est d'élever au maximum le niveau de température dans le lingot à la fin d'une période de temps fixée, en maintenant la température ambiante aussi basse que possible. Reformulé en problème de commande optimale, le problème est résolu numériquement en utilisant la méthode de perturbation pour linéariser les conditions nécessaires pour l'optimisation.

BERECHNUNG DER OPTIMALEN BEHEIZUNG EINER PLATTE MIT TEMPERATURABHÄNGIGER WÄRMELEITFÄHIGKEIT UNTER ANWENDUNG VON FINITEN ELEMENTEN

Zusammenfassung—Eine unendlich lange, feste Platte mit temperaturabhängiger Wärmeleitfähigkeit wird bei Umgebungstemperatur optimal beheizt. Das physikalische Ziel des Problems ist, nach Ende einer festen Zeitperiode ein höheres Temperaturniveau in der Platte zu erreichen, während die Umgebungstemperatur so niedrig als möglich gehalten wird. Nach Umformulierung in ein Regelungsproblem wird dieses unter Anwendung der Finiten-Elementen-Methode numerisch gelöst. Außerdem wird zum Linearisieren der notwendigen Bedingungen für das Optimum als Näherung ein Störungsverfahren angegeben.

АНАЛИЗ ОПТИМАЛЬНОГО НАГРЕВА ПЛИТЫ С ТЕПЛОПРОВОДНОСТЬЮ, ЗАВИСЯЩЕЙ ОТ ТЕМПЕРАТУРЫ, МЕТОДОМ КОНЕЧНЫХ ЭЛЕМЕНТОВ

Аннотация — Производится оптимальный нагрев бесконечной плиты, теплопроводность которой зависит от температуры, при заданной внешней температуре. Исследование предпринято с целью получения более высокого уровня температуры в конце определенного периода времени, в то время как необходимая температура внешней среды сохраняется как можно более низкой. Переформулированная как задача оптимального контроля, она решается численно на основе метода конечных.элементов. Для линеаризации необходимых условий оптимальности также приводится метод малых возмущений.